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NOTES ON INSTANTONS IN TOPOLOGICAL FIELD THEORY AND BEYOND

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Abstract. This is a brief summary of our studies of quantum field theories in a special limit in which the instantons are present, the anti-instantons are absent, and the perturbative corrections are reduced to one-loop. We analyze the corresponding models as full-fledged quantum field theories, beyond their topological sector. We show that the correlation functions of all, not only topological (or BPS), observables may be studied explicitly in these models, and the spectrum may be computed exactly. An interesting feature is that the Hamiltonian is not always diagonalizable, but may have Jordan blocks, which leads to the appearance of logarithms in the correlation functions. We also find that in the models defined on Kähler manifolds the space of states exhibits holomorphic factorization. In particular, in dimensions two and four our theories are logarithmic conformal field theories.

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1. Introduction

Most two- and four-dimensional quantum field theories have two kinds of coupling constants: the actual coupling g, which in particular counts the loops in the perturbative calculations, and the topological coupling, ϑ, the theta-angle, which is the chemical potential for the topological sectors in the path integral. These couplings can be combined into the complex coupling τ and its complex conjugate τ^*. The idea is to study the dependence of the theory on τ, τ^* as if they were two separate couplings, not necessarily complex conjugate to each other.

For example, in the four-dimensional gauge theory one combines the Yang-Mills coupling g and the theta-angle ϑ as follows:

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}. \tag{1.1}$$

For the two dimensional sigma model with the complex target space X, endowed with a Hermitian metric $g_{i\bar{j}}$ and a $(1,1)$ type two-form $B_{i\bar{j}}$ one defines

$$\tau_{i\bar{j}} = B_{i\bar{j}} + i g_{i\bar{j}}. \tag{1.2}$$

If $dB = 0$, then the two-form B plays the role of the theta-angle.

A similar coupling constant may also be introduced in the quantum mechanical model on a manifold X endowed with a Morse function f, with the Lagrangian

$$L = \frac{\lambda}{2}\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + g^{\mu\nu}\partial_\mu f \partial_\nu f\right) - i\vartheta \partial_\mu f \dot{x}^\mu + \dots . \tag{1.3}$$

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where ... denote the possible fermionic terms (in the supersymmetric version). The corresponding coupling τ may then be taken to be

$$\tau = \vartheta + i\lambda.$$

For finite λ the supersymmetric quantum mechanics with the bosonic Lagrangian (1.3) is the model studied by E. Witten in his proof of Morse inequalities [7].

In all these examples we expect the correlators to be functions of τ, τ^*. It is reasonable to expect that they may be analytically continued to the domain of complex couplings g and ϑ. In particular, the theory should greatly simplify in the limit:

$$\tau^* \to -i\infty \ , \qquad \tau \text{ fixed.} \tag{1.4}$$

This is the weak coupling limit, in which the theta-angle has a large imaginary part.

The reason for this simplification is that the theory in this limit is described by a first-order Lagrangian. The corresponding path integral represents the "delta-form" supported on the instanton moduli space, which is essentially *finite-dimensional*. More precisely, the instanton moduli space has components labeled by the appropriate "instanton numbers", and each component is finite-dimensional (after dividing by the appropriate gauge symmetry group). Therefore the correlation functions are expressed as linear combinations of integrals over these finite-dimensional components of the instanton moduli space.

When we move away from the special point $\tau^* = -i\infty$ (with fixed τ), both instantons and anti-instantons start contributing to the correlation functions. The path integral becomes a Mathai-Quillen representative of the Euler class of an appropriate vector bundle over the instanton moduli space, which is "smeared" around the moduli space of instantons (like a Gaussian distribution), see, e.g., [3]. Therefore general correlation functions are no longer represented by integrals over the finite-dimensional instanton moduli spaces and become much more complicated.

In supersymmetric models there is an important class of observables, called the *BPS observables*, whose correlation functions are independent of τ^*. They commute with the supersymmetry charge Q of the theory and comprise the *topological sector* of the theory. The perturbation away from the point $\tau^* = -i\infty$ is given by a Q-exact operator, and therefore the correlation functions of the BPS observables (which are Q-closed) remain unchanged when we move away from the special point. This is the secret of success of the computation of the correlation functions of the BPS observables achieved in recent years in the framework of topological field theory: the computation is actually done in the theory at $\tau^* = -i\infty$, but because of the special properties of the BPS observables the answer remains the same for other values of the coupling constant. But for general observables the correlation functions do change in a rather complicated way when we move away from the special point.

We would like to go *beyond the topological sector* and consider more general correlation functions of non-BPS observables. Why should we be interested in these more general correlation functions? Here are some of the reasons for doing this. Other reasons will become more clear later.

- Understanding non-supersymmetric quantum field theories with instantons: It is generally believed that realistic quantum field theories should be viewed as non-supersymmetric phases of supersymmetric ones. This means that the observables of the original theory may be realized as observables of a supersymmetric theory. But they are certainly not going to be BPS observables. Therefore we need to develop methods for computing correlation functions of such observables.

- Elucidating the pure spinor approach to superstring theory: Non-supersymmetric versions of our models (such as the "curved $\beta\gamma$-systems") play an important role in this approach [1].
- Constructing new invariants: The correlation functions in the topological sector of the quantum field theories considered above give rise to invariants of the underlying manifold, such as the Gromov-Witten and Donaldson invariants. We hope that the correlation functions of the full quantum field theory may allow us to detect some finer information about its geometry.

Since our goal now is to understand the full quantum field theory, and not just its topological sector, it is reasonable to try to describe the theory first for special values of the coupling constants, where the correlation functions are especially simple. It is natural to start with the limit $\tau^* = -i\infty$ (with finite τ). We may then try to extend the results to a neighborhood of this special value by perturbation theory. We hope that this will give us a viable alternative to the conventional approach using the expansion around a Gaussian point. The advantage of this alternative approach is that, unlike in the Gaussian perturbation theory, we do not need to impose a linear structure on the space of fields. On the contrary, the non-linearity is preserved and is reflected in the moduli space of instantons, over which we integrate in the limit $\tau^* \to -i\infty$ (for more on this, see Section 5). This is why we believe that our approach may be beneficial for understanding some of the hard dynamical questions, such as confinement, that have proved to be elusive in the conventional formalism.

The 4D S-*duality* (and its 2D analogue: the mirror symmetry) gives us another tool for connecting our limit to the *physical range of coupling constants*. In a physical theory, in which τ^* is complex conjugate to τ, the S-duality sends $\tau \mapsto -1/\tau$. It is reasonable to expect that S-duality still holds when we complexify the coupling constants τ, τ^*. It should then act as follows:

$$\tau \mapsto -1/\tau, \qquad \tau^* \mapsto -1/\tau^*.$$

Now observe that applying this transformation to $\tau^* = -i\infty$ and finite τ, we obtain $\widetilde{\tau} = -1/\tau$ and $\widetilde{\tau}^* = 0$. These coupling constants are already within the range of physical values, in the sense that both the coupling constant g and the theta-angle ϑ are finite! Therefore we hope that our calculations in the theory with $\tau^* = -i\infty$ could be linked by S-duality to exact non-perturbative results in a physical theory beyond the topological sector.

In [4] (and the forthcoming companion papers [5]) we have launched a program of systematic study of the $\tau^* \to -i\infty$ limit of the instanton models in one, two and four dimensions. In this paper we outline some of the salient features of our constructions and results. We refer to the reader to the above papers for more details and additional references.

2. Lagrangian implementation of the limit $\tau^* \to -i\infty$

In order to study the limit (1.4) properly we pass to the first order formalism (after Wick rotating to Euclidean time and completing the square). Then the Lagrangian (1.3) becomes

$$L \to -ip_\mu \left(\dot{x}^\mu - v^\mu\right) - i\tau \dot{f} + \frac{1}{2\lambda} g^{\mu\nu} p_\mu p_\nu, \tag{2.1}$$

where v^μ is the gradient vector field

$$v^\mu = g^{\mu\nu}\partial_\nu f. \tag{2.2}$$

of the Morse function f.

Let us complexify ϑ and set

$$\tau = \vartheta + i\lambda. \tag{2.3}$$

As $\lambda \to \infty$ with τ fixed (so that $\tau^* \to -i\infty$) the Lagrangian simplifies as follows:

$$L \to L_\infty = -ip_\mu \left(\dot{x}^\mu - v^\mu\right) - i\tau \dot{f}. \tag{2.4}$$

Now, if we integrate the p_μ's out, we immediately see that the path integral localizes onto the union of finite-dimensional moduli spaces $\mathcal{M}_{a,b}$ of the gradient trajectories, i.e. solutions to the equations:

$$\frac{dx^\mu}{dt} = v^\mu, \tag{2.5}$$

obeying the boundary conditions

$$x(-\infty) = a, \qquad x(+\infty) = b.$$

Here a, b are critical points of f, or, equivalently, the zeros of the vector field v.

For the sake of simplicity, let us focus on a supersymmetric model. Then in addition to the bosonic fields $x^\mu(t), p_\mu(t)$ we have their fermionic partners $\psi^\mu(t), \pi_\mu(t)$. The fermionic part of the Lagrangian is, roughly (we skip the couplings to the connections on the tangent bundle etc.),

$$L_{\text{fermion}} = i\pi_\mu \left(\dot{\psi}^\mu - \partial_\nu v^\mu \psi^\nu\right). \tag{2.6}$$

In this case the determinants obtained by integrating over the fluctuations around the solutions to (2.5) cancel, leaving us with the integral over the superspace $\Pi T\mathcal{M}_{a,b}$, where the odd directions come from the solutions to the fermion equations:

$$\frac{d\psi^\mu}{dt} = \partial_\nu v^\mu(x(t))\psi^\nu. \tag{2.7}$$

2.1. **Lagrangian implementation: observables.** The general (local) observables correspond to functions

$$\mathcal{O}(x, p, \pi, \psi)$$

which become differential operators on ΠTX upon quantization. The simplest observables are the so-called *evaluation observables.* They correspond to the functions $\mathcal{O}(x, \psi)$

on ΠTX, i.e., differential forms ϖ on X. Their correlation functions are the easiest to study:

$$
{}_a\langle \mathcal{O}_1(t_1)\dots\mathcal{O}_k(t_k)\rangle_b = e^{-i\tau(f(a)-f(b))}\int_{\mathcal{M}_{a,b}} ev_{t_1}^*\varpi_1\wedge\ldots\wedge ev_{t_k}^*\varpi_k, \tag{2.8}
$$

where

$$
ev_t : \mathcal{M}_{a,b}\to X
$$

is the evaluation of the gradient trajectory at the moment of time t:

$$
ev_t\,[x] = x(t). \tag{2.9}
$$

Note that for closed differential forms ϖ_i such that $d\varpi_i = 0$ the correlator (2.8) is independent of the time positions t_i (at least in the case when X is a Kähler manifold) and defines a simplified version of the celebrated Gromov-Witten invariants.

3. Hamiltonian implementation of the $\tau^*\to -i\infty$ limit

For simplicity let us set temporarily $\tau = 0$ (see formula (2.3)). Therefore $\vartheta = -i\lambda$. The addition of the topological term $-i\vartheta\int df = -\lambda\int df$ to the Lagrangian (see formula (1.3)) amounts to the following redefinition of the wave-functions:

$$
\Psi\mapsto\Psi^{\rm in} = \Psi e^{\lambda f},\qquad \Psi\mapsto\Psi^{\rm out} = \Psi^* e^{-\lambda f}. \tag{3.1}
$$

This maps the standard Hermitian inner product to the pairing

$$
\langle\Psi^{\rm out}|\Psi^{\rm in}\rangle = \int\Psi^{\rm out}\wedge\Psi^{\rm in}.
$$

Once ϑ is allowed to be complex, the manifest unitarity of the usual quantum mechanics is lost, since $\Psi^{\rm out}\neq\left(\Psi^{\rm in}\right)^*$. However, for finite λ we can always undo the transformation (3.1) and establish the isomorphism between the spaces of in- and out-states.

The redefinition (3.1) of the wave-functions leads to the following redefinition of observables:

$$
\mathcal{O}\mapsto\mathcal{O}^{\rm in} = e^{\lambda f}\mathcal{O}e^{-\lambda f},\qquad \mathcal{O}\mapsto\mathcal{O}^{\rm out} = e^{-\lambda f}\mathcal{O}e^{\lambda f}.
$$

In particular, the Hamiltonian $-\frac{1}{2\lambda}\Delta+\frac{\lambda}{2}\|df\|^2$ of the original quantum mechanical model with the Lagrangian (1.3) is mapped to

$$
H^{\rm in}_\lambda = \mathcal{L}_v - \frac{1}{2\lambda}\Delta,\qquad H^{\rm out}_\lambda = -\mathcal{L}_v-\frac{1}{2\lambda}\Delta, \tag{3.2}
$$

where $\mathcal{L}_v$ is the Lie derivative along the gradient vector field v. In the limit $\lambda\to\infty$ they become $H^{\rm in}_\infty = \mathcal{L}_v, H^{\rm out}_\infty = -\mathcal{L}_v$.

More generally, if we wish to keep non-zero $\tau = \vartheta + i\lambda$, then we consider the Hamiltonian

$$
H_{\tau,\tau^*} = \frac{1}{2\lambda}\{d - i\tau df\wedge\ ,\ d^* + i\tau^*\iota_v\}
$$

and then take the limit $\lambda\to+\infty, \vartheta\to -i\infty$, while keeping τ fixed. Then the Hamiltonian tends to

$$
H_\tau = \mathcal{L}_v - i\tau\|v\|^2 = e^{i\tau f}H_\infty e^{-i\tau f}. \tag{3.3}
$$

3.1. **Local theory: harmonic oscillator.** Let us analyze the spectrum of the Hamiltonian (3.3) near a fixed point x_0, $v(x_0) = 0$. The problem reduces to that of the spectrum of harmonic oscillator. The only remaining issue is the effect of the redefinition (3.1) on the well-known eigenstates of the Hamiltonian $H = -\frac{1}{2\lambda}\Delta + \frac{\lambda}{2}\|df\|^2$

There are two basic cases, corresponding to a repulsive critical point, with $f = \omega x^2/2$, and an attractive critical point, with $f = -\omega x^2/2$, (we will assume that $\omega > 0$). In the limit $\lambda \to \infty$ the states and the Hamiltonians are as follows. In the repulsive case (we set $\tau = 0$ for simplicity):

$$\begin{aligned} \Psi^{\text{in}} &= P(x, dx), \qquad \Psi^{\text{out}} = P(\partial_x, \imath_{\partial_x})\delta(x) \\ H^{\text{in}} &= \omega \mathcal{L}_{x\partial_x}, \qquad H^{\text{out}} = -\omega \mathcal{L}_{x\partial_x}. \end{aligned} \tag{3.4}$$

In the attractive case:

$$\begin{aligned} \Psi^{\text{out}} &= P(x, dx), \qquad \Psi^{\text{in}} = P(\partial_x, \imath_{\partial_x})\delta(x) \\ H^{\text{in}} &= -\omega \mathcal{L}_{x\partial_x}, \qquad H^{\text{out}} = +\omega \mathcal{L}_{x\partial_x}, \end{aligned} \tag{3.5}$$

where $P(\cdot)$ is a polynomial differential form. It is easy to see that the spectrum of the Hamiltonian(s) is bounded from below. The negative signs are compensated by the fact that the scaling dimensions of the delta-function and its derivative are also negative. Thus in all cases we get the eigenvalues-values:

$$E_n = |\omega| n \ , \ n = 0, 1, 2, \ldots \ . \tag{3.6}$$

In the non-supersymmetric case there are corrections to the energy levels of them form $\pm\frac{1}{2}\omega$. But in all cases we obtain the real spectrum bounded below. More generally, we could have gotten complex eigenvalues, but their real parts are always bounded below, as is required by stability.

3.2. **Example of a global theory: two dimensional sphere.** The next example illustrates general phenomenon of the state-mixing in the presence of instantons. We study quantum mechanics on $X = \mathbb{S}^2$. Take

$$f = \frac{1}{4}\frac{z\overline{z} - 1}{z\overline{z} + 1}, \qquad g = \frac{dz d\overline{z}}{(1 + z\overline{z})^2}. \tag{3.7}$$

The corresponding gradient vector field

$$v = z\partial_z + \overline{z}\partial_{\overline{z}} \tag{3.8}$$

has two fixed points: $z = 0$ and $z = \infty$.

We can cover X with two coordinate patches, isomorphic to $\mathbb{C}$,

$$\mathbb{C}_0 = \mathbb{S}^2 \backslash \{\infty\}, \qquad \mathbb{C}_\infty = \mathbb{S}^2 \backslash \{0\}.$$

The coordinates z and w on these patches are related via $z = 1/w$. The moduli space of gradient trajectories splits as a disjoint union of the following components:

$$\mathcal{M}_{\infty,\infty} = \{\infty\}, \qquad \mathcal{M}_{\infty,0} \simeq \mathbb{C}^\times, \qquad \mathcal{M}_{0,0} = \{0\}. \tag{3.9}$$

Quantum mechanically, we have a two-well potential with an additional $U(1)$ symmetry. Let us denote the generator of the $U(1)$ rotations by P,

$$P = -i\left(z\partial_z - \overline{z}\partial_{\overline{z}}\right). \tag{3.10}$$

We can lift the degeneracy caused by the $U(1)$ symmetry by studying the common spectrum of the operators $(L_0, \bar{L}_0) = \frac{1}{2}(H + iP, H - iP)$.

What about the two wells? Naively, in the $\lambda \to \infty$ limit the spectrum should be well approximated by the harmonic oscillators corresponding to the two minima of the potential. The function f behaves as

$$f \sim -\frac{1}{4} + \frac{1}{2} z\overline{z} \qquad \text{near} \qquad z = 0$$

(repulsive critical point), and as

$$f \sim \frac{1}{4} - \frac{1}{2} w\overline{w} \qquad \text{near} \qquad z = \infty$$

(attractive critical point). Therefore the analysis of the previous section leads us to the following description of the spaces of "in" states:

$$\begin{aligned} \mathcal{H}^{\mathrm{in}} &= \mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0} \oplus \mathcal{H}^{\mathrm{in}}_{\infty}, \\ \mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0} &= \mathbb{C}[z, \overline{z}, dz, d\overline{z}], \\ \mathcal{H}^{\mathrm{in}}_{\infty} &= \mathbb{C}[\partial_w, \partial_{\overline{w}}, \iota_{\partial_w}, \iota_{\partial_{\overline{w}}}]\delta^{(2)}(w, \overline{w}) dw \wedge d\overline{w}, \end{aligned} \tag{3.11}$$

and "out" states:

$$\begin{aligned} \mathcal{H}^{\mathrm{out}} &= \mathcal{H}^{\mathrm{out}}_{0} \oplus \mathcal{H}^{\mathrm{out}}_{\infty}, \\ \mathcal{H}^{\mathrm{out}}_{\infty} &= \mathbb{C}[w, \overline{w}, dw, d\overline{w}], \\ \mathcal{H}^{\mathrm{out}}_{0} &= \mathbb{C}[\partial_z, \partial_{\overline{z}}, \iota_{\partial_z}, \iota_{\partial_{\overline{z}}}]\delta^{(2)}(z, \overline{z}) dz \wedge d\overline{z}. \end{aligned} \tag{3.12}$$

This gives us the following energy levels:

$$\begin{array}{rclcl} \text{on } \mathcal{H}^{\mathrm{in}}_{\mathbb{C}_0} & : & n + \overline{n} & \text{on} & z^n \overline{z}^{\overline{n}} \\ & & n + \overline{n} + 1 & \text{on} & z^n \overline{z}^{\overline{n}} dz \text{ and } z^n \overline{z}^{\overline{n}} d\overline{z} \\ & & n + \overline{n} + 2 & \text{on} & z^n \overline{z}^{\overline{n}} dz \wedge d\overline{z} \\ \text{on } \mathcal{H}^{\mathrm{in}}_{\infty} & : & n + \overline{n} + 2 & \text{on} & \partial^n_w \partial^{\overline{n}}_{\overline{w}} \delta^{(2)}(w, \overline{w}) \\ & & n + \overline{n} + 1 & \text{on} & \partial^n_w \partial^{\overline{n}}_{\overline{w}} \delta^{(2)}(w, \overline{w}) dw \text{ and } \partial^n_w \partial^{\overline{n}}_{\overline{w}} \delta^{(2)}(w, \overline{w}) d\overline{w} \\ & & n + \overline{n} & \text{on} & \partial^n_w \partial^{\overline{n}}_{\overline{w}} \delta^{(2)}(w, \overline{w}) dw \wedge d\overline{w}. \end{array} \tag{3.13}$$

We have a similar description of $\mathcal{H}^{\mathrm{out}}$.

3.3. **Global theory: problems and their resolution.** Our description (3.11)–(3.13) of the spaces of states raises some serious questions. First of all, the eigenfunctions $z^n \overline{z}^{\overline{n}}$ are not well-defined on the sphere, only on the big cell $\mathbb{C}_0$. Secondly, we have a degenerate spectrum (except for the ground states in the zero- and two-form sectors), and this contradicts the usual expectation that the instantons lift the degeneracy of the spectrum.

It turns out that these problems can be resolved. To begin with, note that we have the limiting wave-functions represented by the delta-functions and their derivatives, which are not functions on the sphere, either. Since we believe that these delta-functions are honest limits of the wave-functions when $\lambda \to \infty$, then we have to allow generalized functions (or distributions) as legitimate wave-functions in this limit. This gives us a hint that we should try to view those wave-functions that are polynomials in z as generalized functions as well. This is possible, but there is a subtlety, which leads to the appearance of Jordan blocks in the Hamiltonian.

So we wish to think of a polynomial $P(z,\overline{z})$ as a distribution on the smooth differential two-forms on $\mathbb{S}^2$. Of course, the pole at infinity makes the naive integral of the product of P and a smooth two-form ω ill-defined (unless ω rapidly decays at ∞). But let us regularize this integral by setting

$$\langle z^n\overline{z}^{\overline{n}},\omega\rangle = \left(\int_{|z|<\epsilon^{-1}} z^n\overline{z}^{\overline{n}}\,\omega\right)_{\epsilon^0}. \tag{3.14}$$

The integral on the right hand side may be written as

$$C_0+\sum_{i>0}C_i\epsilon^{-i}+C_{\log}\log\epsilon+o(1), \tag{3.15}$$

where the C_i's and $C_{\log}$ are some numbers. The right hand side of (3.14) is, by definition, the Hadamard *partie finie* of the above integral, i.e., the constant coefficient C_0 obtained after discarding the terms with negative powers of ϵ and $\log\epsilon$ in the integral (3.14) and taking the limit $\epsilon\to 0$ (this is also reminiscent of the Epstein-Glaser regularization familiar in quantum field theory).

Note that this pairing is not canonical. Should we change ϵ to 2ϵ, the result will change by

$$\log(2)\ \frac{1}{(n-1)!}\frac{1}{(\overline{n}-1)!}\partial_w^{n-1}\partial_{\overline{w}}^{\overline{n}-1}\left(\frac{\omega}{dwd\overline{w}}\right)|_{w=\overline{w}=0},\qquad n,\overline{n}>0.$$

Thus, we cannot separate the monomial $z^n\overline{z}^{\overline{n}}$, considered as a distribution in the above sense, from the delta-like distribution $\frac{1}{(n-1)!(\overline{n}-1)!}\partial_w^{n-1}\partial_{\overline{w}}^{\overline{n}-1}\delta^{(2)}(w,\overline{w})$ (unless $n=0$ or $\overline{n}=0$). Thus, we observe a "mixing" between the states

$$z^n\overline{z}^{\overline{n}}\qquad\text{and}\qquad \frac{1}{(n-1)!(\overline{n}-1)!}\partial_w^{n-1}\partial_{\overline{w}}^{\overline{n}-1}\delta^{(2)}(w,\overline{w}).$$

This is an instanton effect, as one can see clearly from the following calculation.

Let us calculate the correlation function of the following evaluation observables: a function h and a two-form ψ (it is important in this calculation that h is not closed, i.e., is not a BPS observable). We have

$${}_\infty\langle\mathcal{O}_h(0)\mathcal{O}_\psi(t)\rangle_0 = e^{\frac{i\tau}{2}}\int_{\mathbb{C}^\times} h\left(ze^{-t},\overline{z}e^{-t}\right)\psi(z,\overline{z}). \tag{3.16}$$

(the $e^{\frac{i\tau}{2}}$-factor comes from the instanton part of the action, $-i\tau\int_0^\infty df$). The energy spectrum can be extracted by studying the t-dependence of the correlator (3.16). Take, for example, the following function and two-form:

$$h=\frac{1}{1+|z|^2},\qquad \psi=\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}. \tag{3.17}$$

The correlator (3.16) equals

$$(3.18)\qquad {}_\infty\langle \mathcal{O}_h(0)\mathcal{O}_\psi(t)\rangle_0 = e^{\frac{i\tau}{2}}\left(-\frac{1}{1-e^{-2t}} + \frac{2t}{(1-e^{-2t})^2}\right).$$

Naively one would expect the t-dependence of the form:

$$(3.19)\qquad {}_\infty\langle \mathcal{O}_h(0)\mathcal{O}_\psi(t)\rangle_0 = \sum_\alpha e^{-tE_\alpha}\ h_{0,\alpha}\psi_{\infty,\alpha}.$$

where $h_{0,\alpha}$ is the form-factor, the matrix element of the operator h between the vacuum associated to the point 0, and the eigenstate of the Hamiltonian with the energy level E_α, and $\psi_{\infty,\alpha}$ is the form-factor of ψ between this eigenstate and the covacuum associated to the point ∞.

The presence of the t-factor in (3.19) implies that the Hamiltonian is not diagonalizable! Instead, it has Jordan blocks:

$$(3.20)\qquad \exp\left(\ t\cdot\begin{pmatrix} E & e^{\frac{i\tau}{2}} \\ 0 & E\end{pmatrix}\ \right) = \begin{pmatrix} e^{tE} & te^{\frac{i\tau}{2}}e^{tE} \\ 0 & e^{tE}\end{pmatrix}.$$

The reason why we got a Jordan block, as opposed to a matrix with a non-zero entry under the diagonal (which would be diagonalizable with slightly different eigenvalues), is the absence of anti-instantons in our model. If they were present, the anti-instantons would contribute a small matrix element under the diagonal in the Hamiltonian and hence in the evolution operator.

A closer inspection shows that the Hamiltonian $H = z\partial_z + \overline{z}\partial_{\overline{z}}$ acts on the states as follows:

$$(3.21)\qquad H\cdot \partial_w^n\partial_{\overline{w}}^{\overline{n}}\delta^{(2)}(w,\overline{w}) = (n+\overline{n}+2)\partial_w^n\partial_{\overline{w}}^{\overline{n}}\delta^{(2)}(w,\overline{w}),$$

$$(3.22)\qquad H\cdot z^{n+1}\overline{z}^{\overline{n}+1} = (n+\overline{n}+2)\left(z^{n+1}\overline{z}^{\overline{n}+1}\right) + \partial_w^n\partial_{\overline{w}}^{\overline{n}}\delta^{(2)}(w,\overline{w}).$$

The mechanism generating the shift in the last line on (3.22) is the presence of the $\log\epsilon$ terms in the regularized integrals (3.15). Here we assume that $\tau = 0$. For general τ there is a factor $e^{\frac{i\tau}{2}}$ in (3.20). For simply-connected X this factor can always be removed by changing the basis in our space of states. Therefore, τ is not an observable quantity for simply-connected X. In the case of non-simply connected targets (and, in particular, in the 2D sigma models and 4D gauge theories discussed below), the τ-dependence of the Hamiltonian is physical and observable.

3.4. **General Kähler target manifolds.** The above analysis generalizes in a straightforward way to the supersymmetric quantum mechanical models on a Kähler manifold X equipped with a holomorphic vector field ξ coming from a holomorphic torus action on X with a non-empty set of isolated fixed points (see [4]). Under our assumptions, there is a *Bialynicki-Birula decomposition* [2]

$$X = \bigsqcup_{\alpha\in A} X_\alpha$$

of X into complex submanifolds X_α, isomorphic to $\mathbb{C}^{n_\alpha}$, defined as follows:

$$X_\alpha = \{x\in X \mid \lim_{t\to 0}\phi(t)\cdot x = x_\alpha\},$$

where ϕ is the one-parameter subgroup corresponding to ξ. We have the spaces $\mathcal{H}^{\mathrm{in}}$ and $\mathcal{H}^{\mathrm{out}}$ of "in" and "out" states, respectively. The former decomposes as a direct sum

$$\mathcal{H}^{\mathrm{in}} = \bigoplus_{\alpha \in A} \mathcal{H}^{\mathrm{in}}_{\alpha}, \tag{3.23}$$

where $\mathcal{H}_{\alpha}$ be the space of *delta-forms supported on* X_{α}. An example of such a delta-form is the distribution on the space of differential forms on X which is defined by the following formula:

$$\langle \Delta_{\alpha}, \eta \rangle = \int_{X_{\alpha}} \eta|_{X_{\alpha}}, \qquad \eta \in \Omega^{\bullet}(X). \tag{3.24}$$

All other delta-forms supported on X_{α} may be obtained by applying to Δ_{α} differential operators defined on a small neighborhood of X_{α}. The space $\mathcal{H}^{\mathrm{in}}_{\alpha}$ is graded by the degree of the differential form. We have a similar description of $\mathcal{H}^{\mathrm{out}}$.

The Hamiltonian is equal to $\mathcal{L}_{\xi} + \mathcal{L}_{\overline{\xi}}$ plus the sum of off-diagonal terms which may be expressed in terms of Grothendieck–Cousin operators corresponding to adjacent cells in the above decomposition of X. These terms give rise to Jordan blocks in the hamiltonians, as in the case of $\mathbb{S}^2 = \mathbb{CP}^1$ analyzed above.

The correlation functions of these models may be computed both in the Lagrangian approach (as integrals over moduli spaces of instantons) and in the Hamiltonian approach (as matrix elements of operators acting on the space of states). The factorization of the correlation functions over intermediate states then leads to some interesting and non-trivial identities on distributions, which are discussed in detail in [4].

4. Infinite-dimensional versions: two and four dimensions

The two-dimensional sigma model with the target manifold X on the worldsheets of genus zero and one may be analyzed along the same lines as above by interpreting it as a quantum mechanical model on (a covering of) the loop space LX. Similarly, the four-dimensional gauge theory on $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{S}^1$, ..., may be interpreted as quantum mechanics on the space of gauge equivalence classes on the three dimensional sphere. In this section we discuss this briefly. Details will appear in [5].

4.1. **Novikov, Morse-Bott, equivariant Morse...** These theories have important subtleties. The corresponding functions f are multi-valued:

$$f = \int_{\mathbb{S}^1} d^{-1}\omega \qquad \text{in sigma models,}$$
$$f = \int_{\mathbb{S}^3} \mathrm{tr}\left(AdA + \frac{2}{3}A^3 \right) \qquad \text{in YM theory,}$$

so they are really Morse-Novikov functions. They may have non-isolated critical points, like the constant loops in the sigma models, so they are in fact Morse-Bott-Novikov functions. In addition, the above Chern-Simons functional should be viewed as a Morse function in the equivariant sense (due to gauge symmetry of connections). Because of this, some adjustments need to be made in the formalism discussed above. We will not discuss these models in detail here, referring the reader to [4, 5]. We will only give some sample calculations of the correlation functions which show that these models also exhibit logarithmic behavior. This means that the Hamiltonian has Jordan block. For simplicity we consider below twisted models, the twisted $\mathcal{N} = (2, 2)$ sigma models and twisted $\mathcal{N} = 2$ gauge theory in four dimensions, also known as the Donaldson–Witten theory. Again, we stress that we study these models as *full-fledged* supersymmetric quantum field theories, not merely as *topological field theories*.

4.2. **Sigma models.**

4.2.1. *Infinite radius limit.* The twisted two dimensional sigma model with complex target space X is described with the help of the following fields: $X^\mu = (x^i, \overline{x}^{\overline{j}}) : \Sigma \to X$, the momenta $p_{iw}, p_{\overline{j}\overline{w}}$, the scalar fermions $\psi^i, \bar{\psi}^{\overline{j}}$, and their momenta $\pi_{iw}, \bar{\pi}_{\overline{j}\overline{w}}$. The Lagrangian, written in the first order form, reads:

$$
\begin{aligned}
L = &-i\left(p_{iw}\partial_{\overline{w}}x^i + p_{\overline{j}\overline{w}}\partial_w\overline{x}^{\overline{j}} + \pi_{iw}\partial_{\overline{w}}\psi^i + \bar{\pi}_{\overline{j}\overline{w}}\partial_w\bar{\psi}^{\overline{j}}\right)\\
&+ h^{i\overline{j}}p_{iw}p_{\overline{j}\overline{w}} + \frac{1}{2}g_{i\overline{j}}\left(\partial_w x^i\partial_{\overline{w}}\overline{x}^{\overline{j}} - \partial_{\overline{w}}x^i\partial_w\overline{x}^{\overline{j}}\right)\\
&+ \frac{i}{2}\,B_{\mu\nu}\left(\partial_w X^\mu\partial_{\overline{w}}X^\nu - \partial_{\overline{w}}X^\mu\partial_w X^\nu\right) + \text{ fermions.}
\end{aligned}
\tag{4.1}
$$

Upon elimination of the momenta $p_{iw}, p_{\overline{j}\overline{w}}$ the action (4.1) turns into the standard Lagrangian of the type A twisted supersymmetric sigma model with the target space X endowed with the Hermitian metric $g_{i\overline{j}}$ and the B-field $B_{\mu\nu}$ (see [9]).

If the B-field is shifted by the imaginary B-field

$$
B \longrightarrow \tau = B + g_{i\overline{j}}dx^i \wedge d\overline{x}^{\overline{j}}
\tag{4.2}
$$

and the inverse metric $g^{i\overline{j}}$ is sent to zero with τ kept finite (so that $\tau^* \to \infty$), we obtain the Lagrangian (4.1) of the *curved* $\beta\gamma$-bc system, up to the term $\int X^*\tau$. This is a version of the "infinite radius limit" of this sigma model.

Now let us consider the case where $d\tau = 0$ (for X is Kähler this is what we get starting with the model (4.1), for closed B, $dB = 0$). In this case the term $\int X^*\tau$ is topological.

Suppose that X is covered by coordinate patches $X = \cup_\alpha \mathcal{U}_\alpha$. The model with the Lagrangian

$$
L = -i\left(p_{iw}\partial_{\overline{w}}x^i + p_{\overline{j}\overline{w}}\partial_w\overline{x}^{\overline{j}} + \pi_{iw}\partial_{\overline{w}}\psi^i + \bar{\pi}_{\overline{j}\overline{w}}\partial_w\bar{\psi}^{\overline{j}}\right),
\tag{4.3}
$$

restricted to the maps which land in $\mathcal{U}_\alpha$, for some α, is the free $\beta\gamma$-bc system, a $c = 0$ superconformal field theory. Thus, we can relate the chiral algebra of the sigma model in the infinite radius limit to the *chiral de Rham complex* [6].

4.2.2. *Instanton corrections.* In contrast to most of the mathematical literature, we are not interested in this chiral algebra *per se*. Rather, we are interested in the full quantum field theory in the infinite radius limit $\tau^* \to \infty$, in which the chiral and anti-chiral sectors are combined in a non-trivial way.

We claim that just like in the quantum mechanical model, this global theory, i.e., the sigma model with the instanton corrections, has a non-diagonalizable Hamiltonian.

As in the quantum mechanical case, the spectrum of the Hamiltonian can be read off the correlation functions. In the case of the sigma model the Hamiltonian is $L_0 + \bar{L}_0$, the sum of the chiral and anti-chiral Virasoro generators. The Jordan block nature of the Hamiltonian implies that the sigma model (4.3) is a logarithmic conformal field theory (LCFT). Note that the logarithmic corrections to the Virasoro generators have non-perturbative character: they are caused directly by the instantons!

However, we stress that the Hamiltonian is diagonalizable (in fact, is identically equal to zero) on the BPS states. Therefore correlation functions of the BPS observables which have been extensively studied in the literature (and which are closely related to the Gromov-Witten invariants) do not contain logarithms. In order to observe the appearance of logarithms, we must consider non-BPS observables. The Hamiltonian is also diagonalizable on all purely chiral (and anti-chiral) states; thus, the chiral algebra of the theory is free of logarithms.

Let us consider as an example the target manifold $X = \mathbb{CP}^1$. The instantons are labeled by a non-negative integer in this case. The moduli space of degree d instantons $\mathcal{M}_d$ has complex dimension $2d+1$. We consider $d = 1$. Then the corresponding moduli space $\mathcal{M}_1 \simeq PGL_2(\mathbb{C})$. Consider the correlator of the following evaluation observables:

$$\langle \mathcal{O}_{\omega_0}(0)\mathcal{O}_{\omega_\infty}(\infty)\mathcal{O}_{\omega_{\mathrm{FS}}}(1)\mathcal{O}_h(e^{-t})\rangle_{d=1}, \tag{4.4}$$

where

$$\omega_0 = \delta^{(2)}(x)d^2x \ , \quad \omega_\infty = \delta^{(2)}\left(\frac{1}{x}\right)\frac{d^2x}{|x|^4} \ , \quad \omega_{\mathrm{FS}} = \frac{d^2x}{(1+|x|^2)^2},$$

$$h = \frac{1}{1+|x|^2}. \tag{4.5}$$

The delta-function two-forms ω_0, ω_∞, supported at $x = 0$ and $x = \infty$, respectively, reduce the integration over $\mathcal{M}_1$ to that over the locus consisting of the maps:

$$x(w) = Aw \ . \tag{4.6}$$

Thus, (4.4) is equal to:

$$q\int \frac{d^2A}{(1+|A|^2)^2}\frac{1}{1+e^{-2t}|A|^2} \propto q\left(\frac{-1}{1-e^{-2t}} + \frac{2te^{-2t}}{(1-e^{-2t})^2}\right), \tag{4.7}$$

where q is the instanton factor. The t-dependence in (4.7) implies the logarithmic nature of the two dimensional conformal theory, in the same way as in the case of the quantum mechanical models analyzed above. We conclude that the twisted sigma model on $\mathbb{CP}^1$ is a logarithmic conformal field theory.

The space of states of the sigma model may be described in terms of delta-forms supported on the "semi-infinite" strata of a decomposition of the universal cover of $L\mathbb{CP}^1$, similarly to the quantum mechanical models (see Section 3.4). We have a similar description of the sigma models associated to other Kähler manifolds.

4.3. **Logarithmic conformal field theory in four dimensions.** Four-dimensional $\mathcal{N} = 2$ gauge theory with a compact gauge group G can be twisted just like the $\mathcal{N} = (2,2)$ two-dimensional sigma model. The fields of the twisted theory are as follows. The bosons: the gauge field A_m, the complex Higgs field ϕ, $\bar{\phi} = \phi^*$, in the adjoint representation; and the fermions, all in the adjoint representation: the one-form ψ_m, the self-dual two-form χ^+_{mn}, the scalar η. By analogy with the quantum mechanics and the two-dimensional sigma models we introduce a momentum p^+_{mn} – the self-dual two-form valued in the adjoint representation.

The super-Yang-Mills theory can be studied in the limit $g_{\mathrm{YM}} \to 0$, with

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2_{\mathrm{YM}}}$$

kept finite (so that $\tau^* \to -i\infty$). The action of the limiting theory looks as follows:

$$S_{\mathrm{ssdYM}} = \int \mathrm{tr}\left(-ip^+ \wedge F + i\chi^+ \wedge D^+_A\psi + \eta \wedge \star D^*_A\psi + D_A\phi \wedge \star D_A\bar{\phi} + [\psi, \star\psi]\bar{\phi}\right) - \frac{i\tau}{4\pi}\int \mathrm{tr}F\wedge F. \tag{4.8}$$

The action (4.8) makes sense on any manifold $\mathbf{M}^4$. On any $\mathbf{M}^4$ this theory has at least one fermionic symmetry, generated by a scalar supercharge $\mathcal{Q}$ [8]. The usual feature of such a theory is the $\mathcal{Q}$-exactness of the stress-energy tensor, which follows from the fact that all the metric dependence of (4.8) is contained in the $\mathcal{Q}$-exact terms in the Lagrangian.

The conformal invariance of (4.8) is much less appreciated in the physics literature. Let us assume that ϕ is a scalar, degree zero field, while $\overline{\phi}$ and η are half-densities, i.e. transform like $\mathrm{vol}_g^{\frac{1}{2}}$, under the coordinate transformations. Then (4.8) can be rewritten, with explicit metric dependence, as:

$$S_{\mathrm{ssdYM}} = \mathcal{Q}\int \; -ig^{mm'}g^{nn'}g^{\frac{1}{2}}\mathrm{tr}\left(\chi^+_{mn}F_{m'n'}\right) + g^{mn}g^{\frac{1}{4}}\mathrm{tr}\left(\psi_n D_m\overline{\phi}\right) - \frac{1}{4}\mathcal{Q}\int \; g^{mn}g^{\frac{1}{4}}\mathrm{tr}\left(\psi_m\overline{\phi}\right)\partial_n \log g - \frac{i\tau}{4\pi}\int_{\mathbf{M}^4}\mathrm{tr}F\wedge F, \tag{4.9}$$

where $D_m = \partial_m + [A_m, \cdot]$. The first line in (4.9) already defines a nice measure on the space of fields. To make the theory explicitly conformally invariant we modify the stress-energy tensor in a way analogous to the background charge modification of the bosonic free field stress tensor in two dimensions $T \to T + \partial J$, where $J_m \sim \mathcal{Q}\mathrm{tr}\left(\psi_m\bar{\phi}\right)$.

The path integral in the theory (4.8) localizes onto the instanton moduli space of anti-self dual gauge field configurations, i.e., the solutions of the equation

$$F^+_A = 0. \tag{4.10}$$

We now wish to apply our techniques to this gauge theory and demonstrate its logarithmic nature (when we look beyond the topological sector).

Consider the correlation function

$$C(x, y; z) = \langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{S}(z)\rangle, \tag{4.11}$$

where

$$\mathcal{O}(x) = \mathrm{tr}\phi^2 \;, \qquad \mathcal{S}(z) = \mathrm{tr}F_{mn}F^{mn}.$$

We find that

$$C(x, y; z) = \frac{1}{|x-y|^4}\mathcal{C}\left(\frac{(\bar{x}-\bar{y})\cdot(x+y-2z)}{|x-y|^2}\right),$$

where we use the quaternionic notations, $x, y, z \in \mathbb{H}$, and

$$\mathcal{C}(q) \propto \frac{1}{|1-q|^6} \int_0^1 \frac{du}{M^8} \left(M P_4(M) - 3(1+M)^2 (7 + 6M + M^2) \log(1+M) \right), \tag{4.12}$$

where

$$P_4(M) = \frac{1}{5}M^4 + \frac{35}{4}M^3 + 37M^2 + \frac{99}{2}M + 21,$$

$$M = \frac{|1+up|^2}{u(1-u)} = -|p|^2 + \frac{|1+p|^2}{1-u} + \frac{1}{u}, \qquad p = \frac{1+q}{1-q} \in \mathbb{H}.$$

We shall not write down the explicit expression for (4.12). We only mention that it is a sum of rational functions of $|q|^2$, $|p|^2$, $|1-q|^2$ multiplied by logarithms and dilogarithms of $|q|$, $|1-q|$ etc. The logarithms and dilogarithms of q, $1-q$, etc., in (4.12) imply that the four-dimensional theory is a logarithmic conformal field theory. For more details, see [5].

5. New observables

In this section we explain how to extend our analysis to more general observables, such as those corresponding to vector fields and differential operators on the target manifold.

5.1. **The enhancement of the space of observables.** The observables that we have studied so far are mostly the evaluation observables corresponding to differential forms on the target manifold. The novelty of our approach is to consider the pull-backs of *all* differential forms, not necessarily the closed ones (which correspond to the BPS observables comprising the topological sector of the theory). This allowed us to see the previously hidden logarithmic structure of the states and operators in the quantum field theory with instantons. Now we consider an entirely new class of observables; namely, those corresponding to the vector fields and, more generally, differential operators on the target manifold. These observables are invisible in the BPS sector. We now explain how to define these observables in the models at the special point $\tau^* = -i\infty$. One motivation to study them is that the deformation of our models back to finite values of τ^* is achieved using these observables.

The idea is to deform the instanton equations and to study the response of the correlation functions of evaluation observables to this deformation. This dependence of the correlation functions on the deformations may then be interpreted as one caused by the insertion in the correlation function of a new type of observables. Note that the correlation function of closed evaluation observables is independent of the deformations – therefore, it is crucial to include the evaluation observables of non-closed differential forms. It turns out that in this way we may generate the correlation functions of all local observables in the neighborhood of a twisted supersymmetric point in the space of field theories. Thus, we arrive at a perturbative definition of the path integral of our model without violating the intrinsically non-linear structure of the space of fields. We explain in the examples below how this method works for finite-dimensional integrals and in the case of quantum mechanics. A more thorough treatment will be presented in [5].

5.2. **Finite-dimensional case.** Path integral is usually "defined" by a formal extension to the infinite-dimensional case of some procedures that are well-defined for the integrals over finite-dimensional spaces. For example, the well-known Feynman diagram approach to the path integral is based on the relation

$$\int_{\mathbb{R}^n} \mathrm{d}^n\, x\;\exp\left(-\frac{1}{2}(x,Ax)+W(x)+(t,x)\right) = \exp\left(W\left(\frac{\partial}{\partial t}\right)\right)\int_{\mathbb{R}^n}\mathrm{d}^n\, x\;\exp\left(-\frac{1}{2}(x,Ax)+(t,x)\right) \tag{5.1}$$

and the well-known expression for the Gaussian integral on the right hand side. Here we consider $W(x)$ as a polynomial with formal coefficients, $x\in\mathbb{R}^n$, and t belongs to the dual vector space $(\mathbb{R}^n)^*$.

Now we propose to start with another exact relation. Let X be a finite-dimensional manifold, $V \to X$ a vector bundle over X, and v a section of V. Then this relation is

$$\int \mathrm{d}p_a \mathrm{d}\pi_a \mathrm{d}x^i \mathrm{d}\psi^i\;\exp\left(ip_a v^a(x) + i\pi_a\partial_j v^a\psi^j\right)F(x,\psi) = \int_{\text{zeroes of } v}\omega_F \tag{5.2}$$

where ω_F denotes the differential form on X corresponding to the function F on the ΠTX (with even coordinates x^i and odd coordinates ψ^i). The variables p_a and π_a correspond to the even and odd coordinates on V.

Let us now deform v. In other words, let

$$v_\varepsilon = v_0 + \varepsilon^\alpha v_\alpha, \tag{5.3}$$

where v_0 and v_α are section of V, and ε^α are (formal) deformation parameters. Consider the relation (5.2) in which v is replaced by v_ϵ. We will consider the right hand side of (5.2) as the *definition* of the generating function for the integrals of polynomials in the new observables $\mathcal{O}(v_\alpha)$ corresponding to the v_α's. In other words, we define the correlation function

$$\left\langle \mathcal{O}_F e^{\varepsilon^\alpha \mathcal{O}(v_\alpha)}\right\rangle$$

involving the old evaluation observables $\mathcal{O}_F$ corresponding to differential forms and the new ones, $\mathcal{O}(v_\alpha)$, as

$$\int \mathrm{d}p_a \mathrm{d}\pi_a \mathrm{d}x^i \mathrm{d}\psi^i\;\exp\left(ip_a v^a_\epsilon(x) + i\pi_a\partial_j v^a_\epsilon\psi^j\right)\;F(x,\psi) = \int_{\text{zeroes of } v_\varepsilon}\omega_F. \tag{5.4}$$

Now we wish to use the relation (5.4) as the *definition* of the integral on the left hand side in the case when X is infinite-dimensional, provided that the dimension of the space of zeroes of the section v is finite (so that the right hand side of (5.4) makes sense). This is exactly the situation that we encounter in our instanton models. Indeed, the instanton models discussed above (which appear in the limit $\tau^* \to -i\infty$) are described by first order Lagrangians. Therefore the path integral in these models is an infinite-dimensional version of the integral on the left hand side of (5.4). The analogue of formula (5.4) then becomes the statement that the correlation functions in these models localize on the finite-dimensional moduli space of instantons, which is interpreted as the space of zeroes of an appropriate section of a vector bundle over the space of all fields.

Once we have defined the path integral in this way, it is natural to ask: what happens if we deform the instanton moduli space? This means deforming the section v_0, which defines our moduli space, by adding to it $\varepsilon^\alpha v_\alpha$. The point is that the result should be interpreted as the insertion of a new observable $\exp(\varepsilon^\alpha \mathcal{O}(v_\alpha))$ into the path integral (corresponding to the initial instanton moduli space).

Note that in the case when the zeroes of $v_0(x)$ are isolated, the definition (5.4) reproduces the same results as the traditional perturbative approach. However, unlike the traditional approach, our definition does not require that we choose any linear structure on the space X. Therefore it is well-adapted to strongly non-linear systems such as the sigma-models and non-abelian gauge theories.

We will now illustrate how this works in a toy model example of quantum mechanics.

5.3. **Quantum mechanical example.** We take as X the space of maps of the circle $\mathbb{S}^1_t$ (with the coordinate $t \sim t+1$) to another circle $\mathbb{S}^1_q$ (with the coordinate $q \sim q+1$). As the vector bundle V we take the bundle whose fiber at $q(t) : \mathbb{S}^1_t \to \mathbb{S}^1_q$ is

$$V_{q(t)} = \Gamma\left(q^* T\mathbb{S}^1_q \otimes \Omega^1(\mathbb{S}^1_t)\right). \tag{5.5}$$

As a section of this bundle we take:

$$v_0 = (dq - 1 \cdot dt), \tag{5.6}$$

where 1 is understood as a particular vector field ∂_q on the target $\mathbb{S}^1$. We will consider deformations $v_\epsilon = v_0 + \varepsilon^\alpha v_\alpha$, where

$$v_\alpha = 1 \cdot u_\alpha(t) dt, \tag{5.7}$$

with the restriction that

$$\int_0^1 u_\alpha(t) dt = 0.$$

The space of zeroes of the vector field v_ε is the space of solutions of the equation

$$q(t) = q_0 + t + \varepsilon^\alpha \int_0^t u_\alpha(t') dt'. \tag{5.8}$$

For the observable $F = \exp 2\pi i\left(q\left(t_2\right) - q\left(t_1\right)\right) \delta\left(q\left(0\right)\right) \psi(0)$ the integral (5.4) is given by the formula

$$\int \mathcal{D}p(t)\mathcal{D}\pi(t)\mathcal{D}q(t)\mathcal{D}\psi(t) e^{i\int(p(dq-dt)+\pi d\psi)} e^{-i\int\ p\varepsilon^\alpha u_\alpha(t)dt} \cdot \\ \cdot\, e^{2\pi i(q(t_2)-q(t_1))}\delta(q(0))\psi(0) = e^{2\pi i\left(t_2 - t_1 + \varepsilon^\alpha \int_{t_1}^{t_2} u_\alpha(t)dt\right)}. \tag{5.9}$$

Now we take

$$\varepsilon^\alpha u_\alpha(t) = \varepsilon(\delta(t-t_+) - \delta(t-t_-)), \tag{5.10}$$

where

$$t_+ > t_2 > t_- > t_1\ . \tag{5.11}$$

We obtain from (5.9) the following correlation function:[1]

$$\langle e^{\varepsilon p(t_+)} e^{2\pi i q(t_2)} e^{-\varepsilon p(t_-)} e^{-2\pi i q(t_1)} \delta(q(0))\psi(0)\rangle = e^{2\pi i(t_2-t_1)} e^{2\pi i\varepsilon} \tag{5.12}$$

Now, from (5.12) it follows that

$$e^{\varepsilon p} e^{2\pi i q} e^{-\varepsilon p} e^{-2\pi i q} = e^{2\pi i \varepsilon} \ .$$

Thus, we see that the above formalism reproduces the Heisenberg relation in quantum mechanics on a circle. In order to reproduce the Heisenberg relation in its standard form we should consider maps from $\mathbb{R}^1$ to $\mathbb{R}^1$ in a similar fashion.

5.4. **Generalization to two and four dimensions.** Applying a similar formalism in two- and four-dimensional instanton models allows us to introduce new observables in these models and opens the door for a perturbation theory away from the special point $\tau^* = -i\infty$ towards the physical range of the coupling constants.

For example, in two-dimensional sigma models these observables have the form

$$\begin{aligned} \mathcal{O}(v) = {} & V^j(x^i, \overline{x}^{\overline{i}}) p_{jw} + W^{\overline{j}}(x^i, \overline{x}^{\overline{i}}) p_{\overline{j}\overline{w}} \\ & + \psi^i(\pi_{jw}\partial_{x^i} V^j + \pi_{\overline{j}\overline{w}})\partial_{x^i} W^{\overline{j}} + \overline{\psi}^{\overline{i}}(\pi_{jw}\partial_{\overline{x}^{\overline{i}}} V^j + \overline{\pi}_{\overline{j}\overline{w}} \partial_{\overline{x}^{\overline{i}}} W^{\overline{j}}). \end{aligned} \tag{5.13}$$

Hence they correspond to the Lie derivatives with respect to the vector fields

$$v = V^j \partial_{x_j} + W^{\overline{j}} \partial_{\overline{x}^{\overline{j}}}$$

on the target manifold X. In particular, if this vector field is holomorphic, then the corresponding observable belongs to the chiral algebra of the sigma model (the chiral de Rham complex).

Why do we care about including these observables? We would like to understand our models in the vicinity of the special point $\tau^* \to -i\infty$. In the case of sigma models, for example, deformation to finite values of τ^* is achieved by adding to the Lagrangian the term (where $G^{i\overline{j}}$ is a constant matrix)

$$G^{i\overline{j}} \mathcal{O}(e_i)\mathcal{O}(e_{\overline{j}}) \ , \tag{5.14}$$

where e_i, $e_{\overline{i}}$ are the vierbein components, $G^{i\overline{j}} e_i^\mu \otimes e_{\overline{j}}^\nu$ being the inverse metric on X. As (5.14) is bilinear in the operators (5.13), we see that it is the observables of the form (5.13) that are needed in order to define the deformation of our models to the physical range of coupling constants. Implementing this program will allow us to define the correlation functions in our quantum models entirely in terms of finite-dimensional integrals.

6. Remarks and outlook

We now summarize what we have learned so far. In quantum field theories in one, two, and four space-time dimensions we study the limit, in which the loop counting parameter is sent to zero, the theta angle has acquired a large imaginary part, so that a particular combination, the instanton action, is kept finite, while the anti-instanton action is sent to infinity. The resulting theory is solvable and can be used as the

[1] For the general ordering of times, not necessarily agreeing with (5.11), $e^{2\pi i\varepsilon}$ will be replaced by $\exp(2\pi i\varepsilon\, \mathrm{link}([t_+]-[t_-],[t_2]-[t_1]))$, where $\mathrm{link}(A,B)$ is the linking number of the 0-chains A and B.

starting point of a perturbation theory. The simplest models are those in which the path integral measure is canonically defined. Such a theory is obtained from the physical supersymmetric theory with $\mathcal{N} = 2$ supersymmetry by the procedure known as twisting. The resulting theory has a nilpotent symmetry, generated by a scalar supercharge $\mathcal{Q}$, on any worldsheet. However, we look at the full quantum field theory, beyond its topological sector. Let us summarize the salient features of this theory.

6.1. **Logarithmic structure.** Instanton corrections induce mixing between the approximate eigenstates of the energy operator. In the absence of anti-instantons this mixing does not change the energy eigenvalues. It lifts, however, the degeneration of the spectrum by making some of the approximate eigenstates into adjoint vectors of the Hamiltonian, thereby creating Jordan blocks. In order to observe this structure, it is crucial to consider non-BPS observables, i.e. those non-commuting with the supercharge $\mathcal{Q}$. The correlation functions of BPS observables are independent on the worldsheet positions, and probe the vacua of the theory only, on which Jordan blocks (and hence logarithms) cannot arise.

In the quantum mechanics on a Kähler manifold X with the holomorphic $\mathbb{C}^\times$-action, whose $U(1)$ part acts isometrically, the non-diagonal part of the Hamiltonian can be related to the so-called Grothendieck–Cousin operators. They acts on the spaces of delta-like differential forms supported on the strata of our manifold (the ascending and descending manifolds of the Morse function), sending forms on a given stratum to those on the adjacent strata of complex codimension one.

6.2. **Space of states and the fate of Hodge theory.** The supersymmetric quantum mechanical models with finite λ give us a particular realization of the Hodge algebra: a pair of odd operators, acting on a superspace (in this case, differential forms on a manifold), whose anti-commutator is an elliptic even operator with discrete spectrum. As is well-known in the classical examples of Hodge theory, the cohomology of any of these odd operators may be identified with the space of harmonic forms with respect to the elliptic operators. Moreover, if the two odd operators are Hermitian conjugate with respect to some Hermitian pairing on the forms, the space of all forms has an orthogonal decomposition into the exact forms for the first operator, the exact forms for the second operators, and the harmonic forms.

As we take the limit $\lambda \to \infty$, together with the conjugation by $e^{\lambda f}$, the Hodge algebra degenerates to

$$\mathcal{L}_v = \{d,\ \iota_v\} \tag{6.1}$$

Now it is no longer true that the "harmonic" forms (the ground states of the quantum mechanical system in the $\lambda = \infty$ limit) are annihilated by d and ι_v. But they are annihilated by their commutator.

If X is a real manifold and V is a gradient vector field of a general Morse function f, then the typical ground states will be the differential delta-forms Δ_α (see formula (3.24)). These are distributions (or currents) supported on the cells of the Morse cell decomposition. They are the singular differential forms, which are delta-forms in the directions transversal to the corresponding cell, and are constant functions in the directions along the cell. The application of the de Rham differential to such a form produces a delta-form, supported at the boundary of the cell. Thus, the action of de Rham differential on the space of ground states coincides with differential of the Morse complex (note that it is identically equal to 0 in the case of Kähler manifolds, because the cells have even real dimensions). This may be viewed as a reformulation of Witten's approach to Morse theory [7].

We go beyond the ground states and consider other delta-like differential forms supported on the cells of the Morse decomposition. We claim that together they span the space of states of our model (actually, the spaces of "in" and "out" states, which correspond to the descending and ascending cells, respectively). These delta-like forms are interpreted as distributions on our manifold defined by means of an Hadamard-Epstein-Glaser type regularization. Because of the "cutoff" dependence of this regularization, the action of the Hamiltonian becomes non-diagonal. This is how the instanton effects are realized on the space of states in our limit.

6.3. **Holomorphic factorization.** In the quantum mechanical model on Kähler manifolds with $U(1)$ isometry (loop spaces of interest fall into this category) the space of "in" states $\mathcal{H}^{\mathrm{in}}$ has the form

$$\mathcal{H}^{\mathrm{in}} = \bigoplus_{\alpha \in A} \mathcal{H}^{\mathrm{hol}}_{\alpha} \otimes \mathcal{H}^{\text{anti-hol}}_{\alpha}, \tag{6.2}$$

where A is the set of fixed points of the $U(1)$ action. This is a version of holomorphic factorization, which exhibits some familiar features of two-dimensional conformal field theory, such as the appearance of conformal blocks. Note that the decomposition (6.2) is possible because the cells of the Morse decomposition are isomorphic to complex vector spaces in the Kähler case. This implies that the spaces of delta-like forms supported on these cells decompose into tensor products of holomorphic and anti-holomorphic ones. Indeed, a delta-like form supported on a cell may be written as a polynomial differential form on the cell itself times a polynomial in the derivatives in the transversal directions, applied to the delta-form supported on the cell.

We observe a similar decomposition in the two-dimensional sigma models with Kähler target manifolds.

6.4. **New invariants of manifolds?** Going beyond the ground states (i.e., beyond the cohomology of the $\mathcal{Q}$-operator) may lead us to new invariants of four-manifolds. Let us elaborate on this point. The ordinary Donaldson theory produces invariants of the smooth structure of a four manifold $\mathbf{M}^4$ out of the topology of the moduli space of gauge instantons. The latter can be viewed as the investigation of the overlaps of the ground states of the four-dimensional theory. From this perspective, going beyond the ground states is analogous, in a sense, to working with the minimal model of the differential graded algebra of differential forms on a manifold (as opposed to just the cohomology of the manifold). This is reminiscent of D. Sullivan's approach to the reconstruction of the rational homotopy type of smooth manifolds. We therefore hope that the study of the analogous algebras of forms on the moduli spaces of gauge instantons will produce finer invariants of four-manifolds.

6.5. **Non-supersymmetric theories.** The limit $\tau^* \to -i\infty$ may also be studied in the context of non-supersymmetric theories. In this case the definition of the path integral measure requires more work. The spaces of states are defined using half-forms and their infinite-dimensional analogues. In the case of the sigma model on a flag variety one finds an affine Lie algebra at the critical level $k = -h^\vee$ as a chiral symmetry algebra (hence this model is related to the geometric Langlands correspondence). Since these manifolds are not Calabi-Yau, the conformal symmetry is not preserved. Moreover, the analogues of the logarithmic terms in the Hamiltonian generate a mass gap. These models will be studied in Part III of [5].

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